

THE RICCI FLOW ON 2-ORBIFOLDS WITH POSITIVE CURVATURE

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Abstract

The Ricci flow on orbifolds converges asymptotically to a soliton solution. This also provides us with a canonical metric on every orbifold.

0. Introduction

Richard Hamilton [3] proved that under the Ricci flow a metric on any compact 2-dimensional, smooth manifold with positive curvature will converge to one of constant positive curvature. One can extend this result rather easily to 2-dimensional orbifolds whose universal covers are manifolds. In this paper we will prove an interesting result concerning the asymptotic behavior of the Ricci flow on the so-called class of "bad" orbifolds, or orbifolds whose universal cover is not a manifold.

The Main Theorem. *Any metric with positive curvature on a bad orbifold asymptotically approaches a Ricci soliton at time infinity under the Ricci flow, where a soliton is a solution which moves only by diffeomorphism.*

The main theorem gives us the first known example where a non-Kähler-Einstein orbifold converges to a nontrivial Ricci soliton, namely a metric of nonconstant curvature. The main theorem also provides us with a way to get a canonical metric on a bad orbifold. On a compact surface, there are no soliton solutions other than those of constant curvature (see Theorem 10.1 in [3]). Bad orbifolds do not admit metrics of constant curvature, so every soliton solution has nonconstant curvature. The main theorem also suggests strongly that a similar phenomenon may occur on higher dimensional Kähler manifolds.

A local coordinate expression of an equation on a manifold M and on any quotient of M by a finite group action look the same, since an orbifold is locally the quotient of a manifold by a finite group action. It is easy to obtain short time existence for the Ricci flow on an orbifold in

the standard way, and the Harnack's inequality and the evolution equation for Q in [3] hold on any orbifold with positive curvature. Furthermore, by using the evolution equation for Q and a slight modification of the arguments given in [1], the entropy estimate on orbifolds can be easily obtained. The only argument in [1] which fails in the bad orbifold case is the one that guarantees a nonzero lower bound for the injectivity radius. (See [3], p. 251.) Nevertheless, we can complete the proof of the main theorem in the bad orbifold case using Theorems 2.7 and 3.7, which show that every point on an orbifold lies in a ball whose radius is comparable to $1/R_{\max}$ and whose area is comparable to $1/R_{\max}$. Then the main theorem follows directly from the arguments given in [3].

All the 1-gons and 2-gons in this paper are simple unless explicitly stated otherwise.

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1. The topological properties of orbifolds

If a group G acts properly discontinuously on a smooth manifold M , then the quotient space is a smooth orbifold. If the universal cover of a smooth orbifold is a manifold, then we call it a good orbifold; otherwise, we call it bad.

Theorem 1.1. *The only bad 2-dimensional orbifolds without boundary are of the following types:*

(a) Z_p -teardrop $T: S^2$ with an orbifold point with an angle $2\pi/p$. Its Euler characteristic is $\chi(T) = 1 + 1/p$; p is an integer.

(b) (Z_p, Z_q) -football $F: S^2$ with two orbifold points with angles $2\pi/p$ and $2\pi/q$. Its Euler characteristic is $\chi(F) = 1/p + 1/q$; p and q are distinct integers, not less than 2.

(See [4], Theorem 2.3.)

Proof. See [4].

For more details about the topological properties of orbifolds see [4].

2. The geometric properties of Z_p -teardrops with positive curvature

From now on we consider M a compact orbifold whose scalar curvature satisfies $0 < R \leq R_{\max}$. We consider a geodesic 2-gon a broken loop consisting of two geodesic segments.

Lemma 2.1. *Let $D^2 \subset M$ be a topological 2-disc, not containing any orbifold points in the interior, whose boundary is a convex geodesic 2-gon. Then $L[\partial D^2] \geq 2\pi/\sqrt{R_{\max}}$.*

Proof. Let γ be the shortest geodesic 2-gon in D^2 . We may assume $L[\gamma] < 2\pi/\sqrt{R_{\max}}$, and we will derive a contradiction. This implies A cannot be conjugate to B along γ (see Corollary 1.30 in [1]). We will imitate the variational arguments in Lemma 5.6. in [1] to show that γ can be shortened. If γ makes angles at two points, call them A and B . If γ makes an angle at A , choose B to divide γ in half. If γ makes no angles, choose A and B again to divide γ in half. Let γ be the union of the two geodesic segments γ_1 and γ_2 .

Case 1. Assume $A \in \text{int}(D^2)$ and $B \in \text{int}(D^2)$. (See Figure 1, next page.)

Then all of $\gamma \in \text{int}(D^2)$. By the variation arguments in Lemma 5.6 in [1], we know $\angle A = \angle B = \pi$. This tells us that γ is a closed geodesic loop in D^2 . From the fact that D^2 is an even-dimensional manifold we get A is conjugate to B along γ (see Theorem 5.9 (1) in [1]). By Corollary 1.30 in [1], we get $L[\partial D^2] \geq 2\pi/\sqrt{R_{\max}}$.

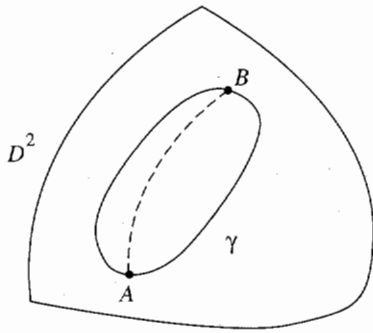
Case 2. Assume $A \in \partial D^2$.

(I) Assume $B \in \text{int}(D^2)$. Then $\angle B = \pi$, as in Case 1. (See Figure 2, next page.)

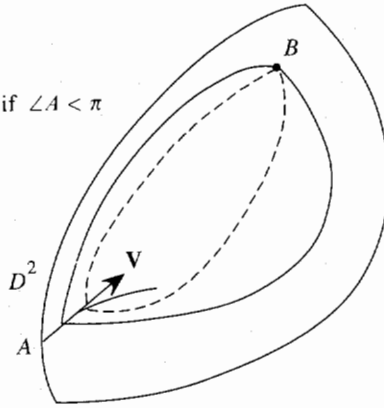
γ_1 and γ_2 cannot have any arc on the boundary. If γ_i had an arc on the boundary, then we would have $\gamma_i \subset \partial D^2$ since γ_i and ∂D^2 are geodesics. This tells us that B is on the boundary. ∂D^2 is a convex geodesic 1-gon so we have $\angle A < \pi$. From Lemma 5.6 in [1], we can find some other shorter 2-gons in D^2 . Thus a shortest 2-gon cannot occur in this case.

(II) Assume $B \in \partial D^2$. (See Figure 3, p. 579.)

∂D^2 is a convex geodesic 2-gon, so we have $\angle A \leq \pi$ and $\angle B \leq \pi$. If $\angle A = \angle B = \pi$, then γ is a closed geodesic. By Corollary 1.30 in [1], we have $L[\partial D^2] \geq 2\pi/\sqrt{R_{\max}}$. If either $\angle A < \pi$ or $\angle B < \pi$ (see Lemma 5.6 in [1]), we can find some other shorter 2-gons in D^2 with at least one endpoint in the interior of D^2 . So the endpoints of the shortest 2-gon cannot both be on the boundary.



if $\angle A < \pi$



pick a tangent vector V at A
 $\rightarrow \angle A = \angle A_1 + \angle A_2$,
 where $\angle A_1 < \pi/2$,
 $\angle A_2 < \pi/2$.
 (See Lemma 5.6 in [1].)

FIGURE 1

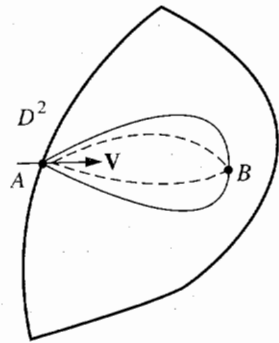
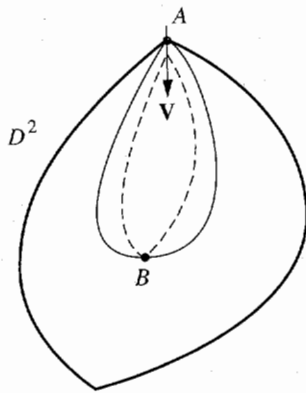


FIGURE 2

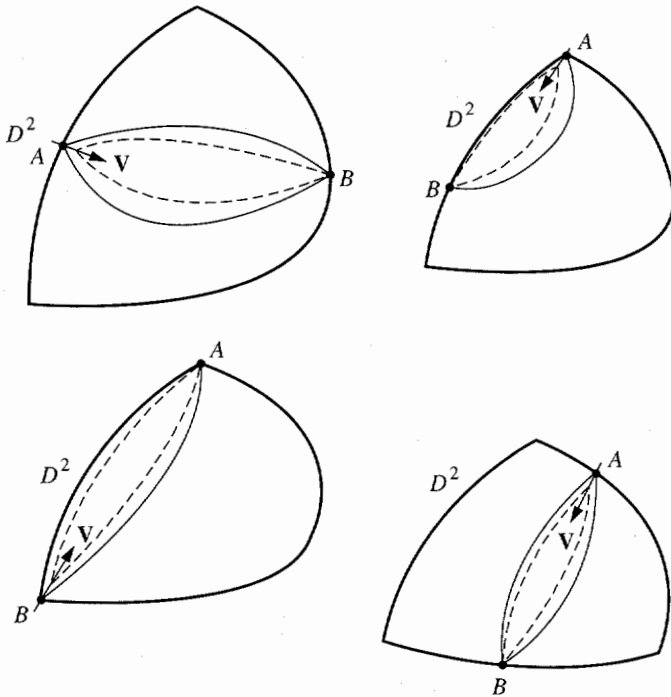


FIGURE 3

Suppose now that M is a teardrop orbifold with one cone point at P in the rest of this section. Let $U = M - P_0$, where P_0 satisfies $d(P_0, P) = \max\{d(P, m) \mid m \in M\}$.

Corollary 2.2. *Let M be a Z_p -teardrop with positive curvature. Denote by $i(P, U^*)$ the injectivity radius of the orbifold point P in the universal cover U^* of U , where $\Pi^*: U^* \rightarrow U$ is the covering map at P . Then*

$$(2.1) \quad i(P, U^*) \geq \frac{\pi}{\sqrt{R_{\max}}}.$$

Proof. Suppose $d(P_0, P) \geq i(P, U^*)$. We may assume that $i(P, U^*) < \pi/\sqrt{R_{\max}}$. Then (Lemma 5.6 in [1]) we have a shortest geodesic 1-gon γ_1 at P on U^* . The metric on U^* is invariant under the Z_p -action. So we have p shortest geodesic 1-gons $\{\gamma_i\}_{i=1}^p$ in U^* , where $L[\gamma_i] \leq 2i(P, U^*) < 2\pi/\sqrt{R_{\max}}$ for all i , and $\gamma_i \cap \gamma_j = P$, for any distinct i and j . Since the exponential map is injective within the injectivity radius and $L[\gamma_i] \leq 2i(P, U^*)$, the disc enclosed by each γ_i has an angle less than $2\pi/p$ at P . Then we have $\Pi^*(\gamma_i) = \gamma$ in M , $\forall i$.

From the topology of M , γ cuts M into two 2-discs $\{D_j\}_{j=1}^{j=2}$. $P \in \gamma$, so D_j is a smooth 2-disc and P is not in the interior of D_j , for both j . The total angle at P is $2\pi/p < \pi$. (See Theorem 1.1.) So ∂D_j is a convex, geodesic 1-gon for both j . By Lemma 2.1 we have $2 \cdot i(P, U^*) = L[\gamma_j] \geq 2\pi/\sqrt{R_{\max}}$. That is a contradiction. So we have

$$(2.2) \quad i(P, U^*) \geq \frac{\pi}{\sqrt{R_{\max}}}.$$

Suppose $d(P_0, P) < i(P, U^*)$. $\Pi^*(\partial U^*) = P_0$ in M and every point on ∂U^* has a unique shortest geodesic from P . This implies that there are many geodesic 2-gons at P and P_0 ; let η be one of them. The cone angle at P is less than π , and η cuts M into two 2-discs $\{D_k\}_{k=1}^2$. The boundary of one of the two discs is a convex geodesic 2-gon. By Lemma 2.1, we have $L[\eta] \geq 2\pi/\sqrt{R_{\max}}$, which implies $i(P, U^*) > d(P_0, P) \geq \pi/\sqrt{R_{\max}}$.

On the universal cover U^* of U , $B_r^* = \exp_p^*(B(0, (r\pi/\sqrt{R_{\max}})))$ is strongly convex if $r < \frac{1}{2}$. (See Theorem 5.14 in [1].) Let $B_r = \Pi^*(B_r^*)$, where $\Pi^*: U^* \rightarrow U$ is the p th covering map at P .

Lemma 2.3. *Any geodesic η in $N = M - B_{1/4}$ which starts at ∂N and returns to ∂N must have length $L[\eta] \geq 3\pi/(2\sqrt{R_{\max}})$.*

Proof. (See Figure 4.) Let γ be the shortest geodesic in N which starts and ends at ∂N . Either γ passes through P_0 or it does not. If γ does not pass through P_0 , then we can lift γ to U^* . Since $B_{1/4}^*$ is strongly convex, γ cannot have length zero. If γ does pass through P_0 , then since P_0 is the furthest point from P , γ cannot have length zero. Let $a, b \in M$ denote the endpoints of the shortest geodesic γ . A variational argument easily shows that γ must intersect ∂N at right angles. The Gauss lemma tells us that $\forall \xi \in \partial B_{1/4}$, the unique minimizing geodesic γ_ξ from p to ξ intersects $\partial B_{1/4}$ at a right angle. Hence $\gamma_a \cup \gamma_b \cup \gamma$ is a geodesic 1-gon at the orbifold point. By Lemma 2.1, $L[\gamma_a \cup \gamma_b \cup \gamma] \geq 2\pi/\sqrt{R_{\max}}$.

From the definition of $B_{1/4}$, we have $L[\gamma_a] = L[\gamma_b] = \pi/(4\sqrt{R_{\max}})$, which implies $L[\gamma] \geq 3\pi/(2\sqrt{R_{\max}})$.

Lemma 2.4. *Any geodesic 1-gon γ in the closure of $B_{1/4} - B_{1/8}$ with the endpoint $x \in \partial B_{1/4}$ has length $\geq C_1/\sqrt{R_{\max}}$, where*

$$(2.3) \quad C_1 = \min \left\{ 2\pi, \frac{2\sqrt{2}\pi}{p} \sin \left(\frac{\sqrt{2}\pi}{16} \right) \right\}.$$

Proof. (See Figure 5, p. 582.) Let $D \supset B_{1/4}$ be the disc which is enclosed by γ . If D does not contain the orbifold point P , then by

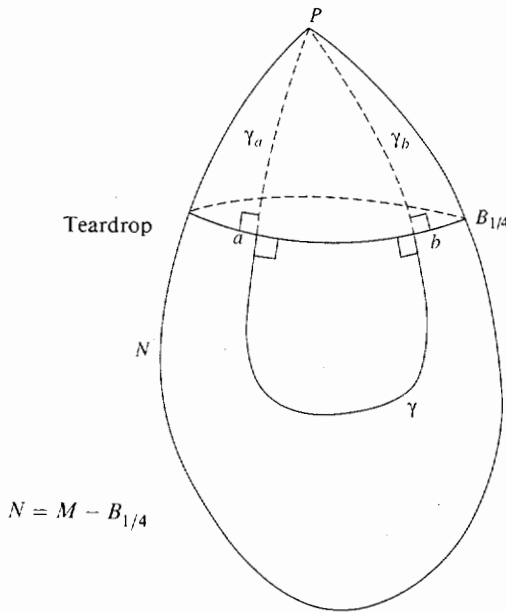


FIGURE 4

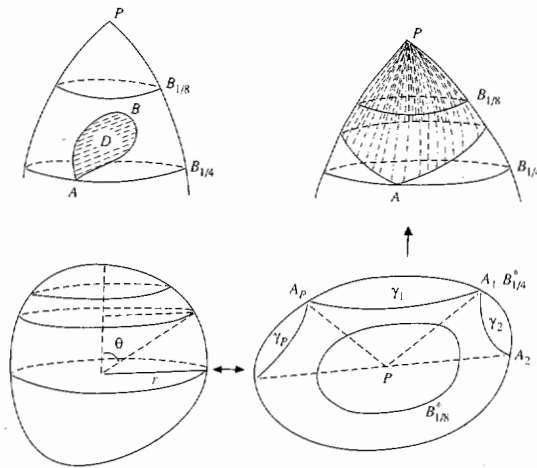
Lemma 2.1, $L[\gamma] \geq 2\pi/\sqrt{R_{\max}}$. If D does contain P , then consider the lifting γ^* of γ in the universal cover $B_{1/4}^*$ of $B_{1/4}$. The fact that D contains P tells us that the endpoints of γ^* are two different preimages of x . Then by Corollary 1.30 in [1] we may compare $B_{1/4}^*$ with the standard sphere of constant curvature R_{\max} . This implies

$$L[\gamma] \geq \frac{2\sqrt{2}\pi \sin\left(\frac{\sqrt{2}\pi}{16}\right)}{p\sqrt{R_{\max}}}$$

Lemma 2.5. *If γ is the shortest geodesic 2-gon in M with at least one endpoint in $N = M - B_{1/4}$, then $L[\gamma] \geq C_2/\sqrt{R_{\max}}$, where*

$$C_2 = \min \left\{ 2\pi, \frac{3\pi}{2}, \frac{2\sqrt{2}\pi}{p} \sin\left(\frac{\sqrt{2}\pi}{16}\right), \frac{\pi}{4} \right\}.$$

Proof. (See Figure 6, p. 583.) If γ makes angles at two points, call them A and B . If γ makes an angle at A , choose B to divide γ in half. If γ makes no angles, choose A and B again to divide γ in half. Let γ be the union of the two geodesic segments, γ_1 and γ_2 .



the standard sphere with radius $r = \sqrt{2/R_{\max}}$
 $L[\partial B_i] = 2\pi r \sin \theta$.
 $l\pi/\sqrt{R_{\max}} = r\theta = \sqrt{2}/\sqrt{R_{\max}}\theta$.
 so $\theta = l\pi/\sqrt{2}$.
 $L[\cup_{i=1}^p \gamma_i] \geq \min_{(1/8) \leq i \leq 1/4} 2\pi \cdot \frac{\sqrt{2}}{\sqrt{R_{\max}}} \sin(l\pi/\sqrt{2}) \geq (2\sqrt{2}\pi)/\sqrt{R_{\max}} \sin((\sqrt{2}\pi)/16)$.
 so $L[\gamma] \geq (2\sqrt{2}\pi)/(P\sqrt{R_{\max}}) \sin((\sqrt{2}\pi)/16)$.

FIGURE 5

Case 1. Assume $A \in \text{int}(N)$. By Lemma 5.6 in [1], we have $\angle A = \pi$. Any closed curve which connects $\partial B_{1/4}$ and $\partial B_{1/8}$ has length larger than $(\pi/4)\sqrt{R_{\max}}$. Since $C \geq \pi/4$, we may only consider the case where $B \in \text{int}(M - B_{1/8})$, which implies $\angle B = \pi$. Then we get a closed geodesic γ . By Case 1 in Lemma 2.1, we have

$$(2.4) \quad L[\gamma] \geq \frac{2\pi}{\sqrt{R_{\max}}}.$$

Case 2. Assume $A \in \partial(N)$ and $B \notin \text{int}(N)$.

(I) Assume $B \notin \partial B_{1/4}$. Since $C \geq \pi/4$ we may only consider the case where γ is in $B_{1/4} - B_{1/8}$. Then $\angle B = \pi$ (see Lemma 5.6 in [1]) and γ is a geodesic 1-gon. By Lemma 2.4, we have $L[\gamma] \geq C_1/\sqrt{R_{\max}}$, where

$$(2.5) \quad C_1 = \min \left\{ 2\pi, \frac{2\sqrt{2}\pi}{p} \sin \left(\frac{\sqrt{2}\pi}{16} \right) \right\}.$$

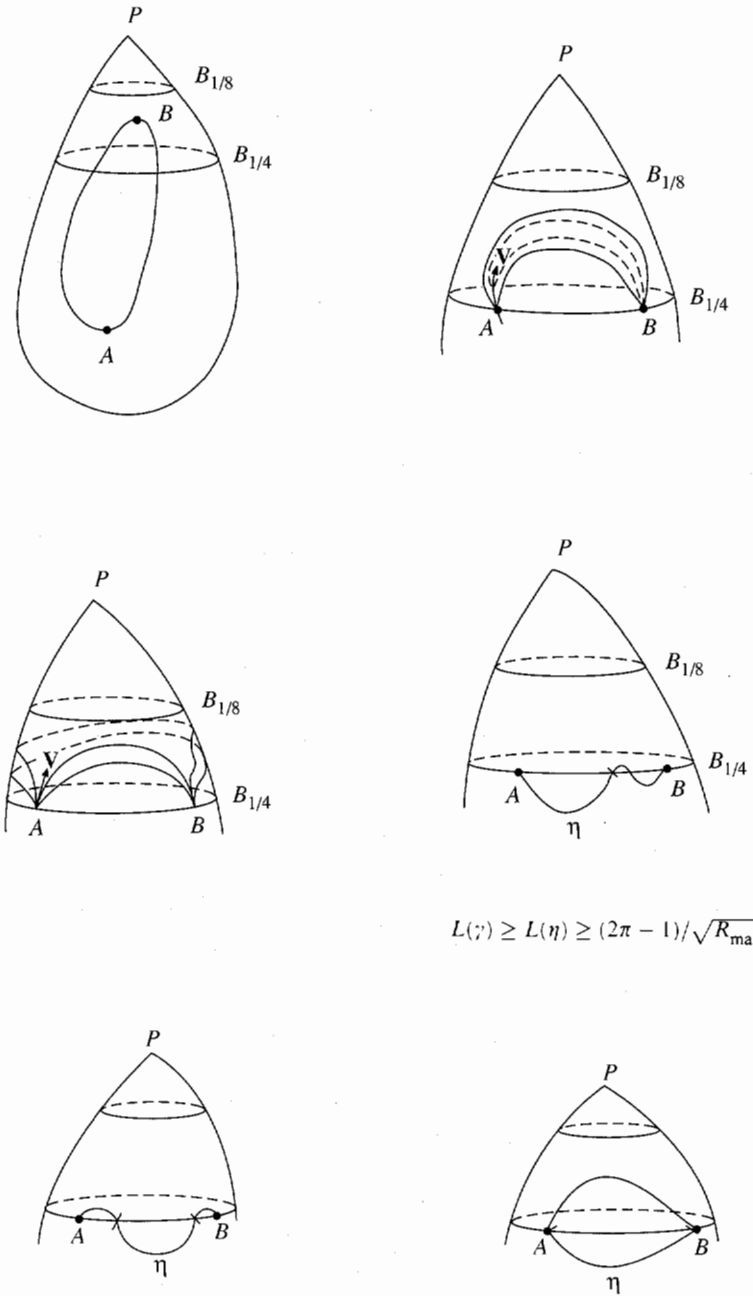


FIGURE 6

(II) Assume $B \in \partial B_{1/4}$. If γ has an arc η in N , then η is a geodesic segment which starts and ends at ∂N . By Lemma 2.3, we have

$$(2.6) \quad L[\gamma] \geq L[\eta] \geq \frac{3\pi}{2} \sqrt{R_{\max}}.$$

If γ_1 and γ_2 are both in $B_{1/4}$, we have $\angle A \leq \pi$ and $\angle B \leq \pi$. Using the same arguments given in Lemma 2.1, we get

$$(2.7) \quad L[\gamma] \geq \frac{2\pi}{\sqrt{R_{\max}}}.$$

Corollary 2.6. For each point ξ in N , we have $i(\xi, M) \geq C_2/(2\sqrt{R_{\max}})$, where

$$(2.8) \quad C_2 = \min \left\{ 2\pi, \frac{3\pi}{2}, \frac{2\sqrt{2}\pi}{p} \sin \left(\frac{\sqrt{2}\pi}{16} \right), \frac{\pi}{4} \right\}.$$

Theorem 2.7. If M is any teardrop with positive curvature, then for each point $\xi \in M$, we have

$$A \left(\exp_{\xi} \left(B \left(0, \frac{\pi}{\sqrt{R_{\max}/2}} \right) \right) \right) \geq C/R_{\max},$$

where

$$(2.9) \quad C = \min \left\{ 2(1 - \cos C_2)[C_2]^2\pi, \frac{2\pi(1 - \cos 1/4)}{8p} \right\}.$$

Proof. Let H be a 2-dimensional Riemannian manifold with curvature $R_{\max} \geq R_H$. For any point ξ on H , assume $\exp_{\xi}|_{B+(0, \rho/\sqrt{R_{\max}/2})}$ is an imbedding and $\pi \geq \rho$. Then, if we compare it with the standard 2-sphere with constant scalar curvature R_{\max} , by Corollary 1.30 in [1], we have

$$(2.10) \quad L \left(\exp_{\xi} \left(\partial B \left(0, \frac{\nu}{\sqrt{R_{\max}/2}} \right) \right) \right) \geq \frac{2\pi(\sin \nu)}{\sqrt{R_{\max}/2}}, \quad \text{for all } \nu \geq \rho.$$

In particular, we have

$$(2.11) \quad A \left(\exp_{\xi} \left(B \left(0, \frac{\rho}{\sqrt{R_{\max}/2}} \right) \right) \right) \geq 2(1 - \cos \rho)\pi\rho^2/R_{\max}.$$

Theorem 2.7 follows easily from this result as we will now demonstrate.

If $\xi \in$ the closure of N , then by Corollary 2.6, we have

$$(2.12) \quad \begin{aligned} & A \left(\exp_{\xi} \left(B \left(0, \frac{\pi}{\sqrt{R_{\max}/2}} \right) \right) \right) \\ & \geq A \left(\exp_{\xi} \left(B \left(0, \frac{C_2}{\sqrt{R_{\max}/2}} \right) \right) \right) \\ & \geq 2(1 - \cos C_2)[C_2]^2\pi/4R_{\max}. \end{aligned}$$

If ξ is in $B_{1/4}$, then

$$(2.13) \quad \exp_P \left(B \left(0, \frac{1}{4\sqrt{R_{\max}/2}} \right) \right) \subset \exp_\xi \left(B \left(0, \frac{1}{2\sqrt{R_{\max}/2}} \right) \right).$$

On the universal covering U^* of U ,

$$A \left(\exp_P^* \left(B \left(0, \frac{1}{4\sqrt{R_{\max}/2}} \right) \right) \right) \geq \frac{\pi}{8R_{\max}},$$

which implies

$$(2.14) \quad \begin{aligned} A \left(\exp_\xi \left(B \left(0, \frac{\pi}{\sqrt{R_{\max}/2}} \right) \right) \right) &\geq A \left(\exp_\xi \left(B \left(0, \frac{1}{2\sqrt{R_{\max}/2}} \right) \right) \right) \\ &\geq A \left(\exp_\xi \left(B \left(0, \frac{1}{4\sqrt{R_{\max}/2}} \right) \right) \right) \\ &\geq \frac{2\pi(1 - \cos 1/4)}{8pR_{\max}}. \end{aligned}$$

Let $C = \min \left\{ 2(1 - \cos C_2)[C_2]^2\pi, \frac{2\pi(1 - \cos 1/4)}{8p} \right\}$.

3. The geometric properties of footballs with positive curvature

In this section we assume that M is a football orbifold with cone angles $2\pi/p$ and $2\pi/q$, and with positive curvature. In order to apply the same techniques for teardrops to footballs, first, we need to have some control over the lower bound of the distance between the two orbifold points, $d(P, Q)$.

Theorem 3.1. *On a (Z_p, Z_q) -football M with positive curvature, we have*

$$(3.1) \quad d(P, Q) \geq \frac{\pi}{\sqrt{R_{\max}}},$$

where P, Q are the orbifold points.

Proof. Let P_0 be a point which satisfies

$$d(P_0, P) = \max\{d(P, m) \mid m \in M\}.$$

Case 1. Assume $d(P, Q) = d(P_0, P)$. Let $V = M - Q$. Consider the p th covering $\Pi: V^* \rightarrow V$ with respect to the Z_p -action at P . Since $V = M - Q$, we have $\Pi(\partial V^*) = Q$. This implies $d(\partial V^*, P) \geq d(P, Q)$. Let $i(P, V^*)$ denote the injectivity radius of P in V^* .

(I) Assume $i(P, V^*) \leq d(P, Q)$. Combining Corollary 1.30 and Lemma 5.6 in [1], either $i(P, V^*) \geq \pi/\sqrt{R_{\max}}$ or there is a shortest 1-gon γ at P in $V^* \cup \partial V^*$ with $L[\gamma] = 2i(P, V^*)$. The metric on $V^* \cup \partial V^*$ is invariant under the Z_p -action, and the exponential map is injective within the injectivity radius; again, we get p disjoint geodesic 1-gons. So if γ exists, then the disc enclosed by γ must have an angle less than $2\pi/p$. Since $2\pi/p \leq \pi$, the geodesic 1-gon γ is convex. By Lemma 2.1, we have $L[\gamma] \geq 2\pi/\sqrt{R_{\max}}$ and $i(P, V^*) \geq \pi/\sqrt{R_{\max}}$. So we have $d(P, Q) \geq \pi/\sqrt{R_{\max}}$.

(II) Assume $i(P, V^*) > d(P, Q)$. This tells us that

$$\max\{d(y, P) \mid y \in \partial V^*\} = d(P, Q).$$

Otherwise, there would be a $y \in V^*$ with $i(P, V^*) \geq d(y, P) > d(P, Q)$, which contradicts the assumption $d(P, Q) = d(P_0, P) = \max\{d(P, m) \mid m \in M\}$. So we have

$$\Pi \left\{ \partial \exp_p^*(B(0, d(P, Q))) \right\} = Q.$$

This also implies that there are 2-gons $\gamma \subset M$ at P and Q with $L[\gamma] = 2d(P, Q)$. Any 2-gon $\gamma \subset M$ at P and Q must enclose a disc D_γ , where ∂D_γ is a convex geodesic 2-gon since again the angles at P and Q are $\leq 2\pi/p$ and $2\pi/q$ which are $\leq \pi$. By Lemma 2.1, we have

$$(3.2) \quad 2d(P, Q) = L[\gamma] \geq \frac{2\pi}{\sqrt{R_{\max}}}.$$

This implies $d(P, Q) \geq \pi/\sqrt{R_{\max}}$.

Case 2. Assume $d(P, Q) < d(P_0, P)$. Let $U = M - P_0$. Consider the q th covering U_q of U with respect to the Z_q -action at Q , where $\Pi_q: U_q \rightarrow U$. Fix a preimage of P , namely P_1 , and consider the p th covering U_p^* of U_q with respect to the Z_p -action at P_1 , where $\Pi_p^*: U_p^* \rightarrow U_q$. Let $i(P_1, U_p^*)$ denote the injectivity radius of P_1 in U_p^* . Let P_k be any preimage of P in U_q other than P_1 . Let ω be the minimizing geodesic connecting P and P_k . If $Q \in \omega$, then $\Pi_q(\omega)$ is a convex 2-gon at P, Q in M . If $Q \notin \omega$, then $\Pi_q(\omega)$ is a convex 1-gon at P in M . In both cases, by Lemma 2.1, we have $L[\omega] \geq (2\pi)/\sqrt{R_{\max}}$. Thus

$$(3.3) \quad d(P, P_k) \geq \frac{2\pi}{\sqrt{R_{\max}}}.$$

(I) Assume $d(P, Q) < i(P_1, U_p^*)$. (See Figure 7.) $d(P, Q) < i(P_1, U_p^*)$, so there exists a unique minimizing geodesic γ in M connecting P and Q . Choose the unit tangent vector V at Q in M such that the two angles

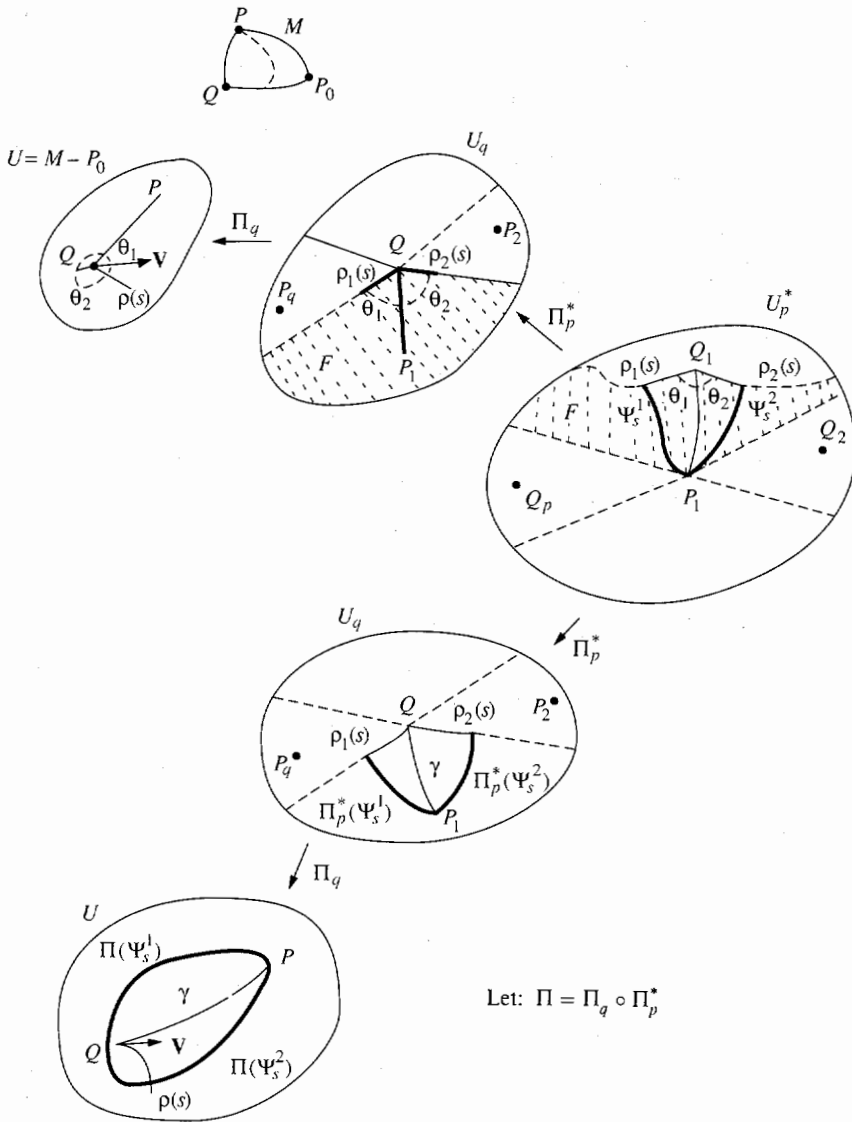


FIGURE 7

θ_1 and θ_2 formed by V and $\gamma'(Q)$ are equal. We also may assume that $q > p \geq 2$, so $\theta_1 = \theta_2 = \pi/q \geq \pi/3 < \pi/2$. Let $\rho(s)$ be the geodesic which starts at Q in the direction of V and is parameterized by the arc

length s . Let F be a connected fundamental domain of Π_q in U_q such that $\Omega \cap \partial F \subset \Pi_q^{-1}(\rho)$, where Ω is a small neighborhood of Q . Since F is the fundamental domain, each $\rho(s)$ in $\Pi_q(\Omega)$ has two different liftings, $\rho_1(s)$ and $\rho_2(s)$, on the closure of F . Π_p^* is a p th covering map at P_1 , so we may identify the closure of F as a subset of U_p^* . Using $d(P, Q) < i(P_1, U_p^*)$, $\theta_1 = \theta_2 = \pi/q < \pi/3 < \pi/2$, and the first variational formula in [1], for small $s > 0$, there is a unique minimizing geodesic arc Ψ_s^i in F connecting P and $\rho_i(s)$ for $i = 1, 2$ with

$$L[\Psi_s^i] < d(P, Q),$$

$$\Psi_s^1 \cap \Psi_s^2 = \emptyset,$$

and

$$(3.4) \quad \Pi_p^*[\Pi_q(\Psi_s^1)] \cap \Pi_p^*[\Pi_q(\Psi_s^2)] = \{p, \rho(s)\}.$$

So $\varphi = \Pi_p^*[\Pi_q(\Psi_s^1 \cup \Psi_s^2)]$ is a 2-gon in M , with

$$(3.5) \quad L[\Psi_s^1] + L[\Psi_s^2] = L[\varphi] < 2d(P, Q).$$

This implies $Q \notin \varphi$ and the shortest 2-gon ω at P in M does not pass Q . So ω is a 1-gon. Then, by Lemma 2.1,

$$L[\varphi] \geq L[\omega] \geq \frac{2\pi}{\sqrt{R_{\max}}},$$

which gives

$$(3.6) \quad d(P, Q) \geq \frac{\pi}{\sqrt{R_{\max}}}.$$

(II) Assume $d(P, Q) \geq i(P_1, U_p^*)$. Without loss of generality, we may assume $i(P_1, U_p^*) < (2\pi)/\sqrt{R_{\max}}$. Then $\exp_{P_1}\{B\{0, i(P_1, U_p^*)\}\}$ is a smooth disc without touching any preimage of P other than P_1 . From $d(P, Q) \geq i(P_1, U_p^*)$, we can find p shortest geodesic 1-gons $\{\delta\}$ in U_p^* with $i(P_1, U_p^*) = (L[\delta])/2$. The exponential map is injective within the injectivity radius, so the p shortest 1-gons $\{\delta\}$ cannot intersect each other except at the point P . Particularly the disc in U^* , which is enclosed by one of the 1-gons $\{\delta\}$, has an angle less than $2\pi/p \leq \pi$ at P . By Lemma 2.1, we have

$$(3.7) \quad L[\delta] \geq \frac{2\pi}{\sqrt{R_{\max}}},$$

which implies $d(P, Q) \geq \pi/\sqrt{R_{\max}}$.

Theorem 3.2. *If M is any football with positive curvature, then the injectivity radius of the orbifold point P (or Q) in the universal cover of $M - Q$ (or $M - P$) is greater than $\pi/\sqrt{R_{\max}}$. Particularly we have*

$$(3.8) \quad A \left(\exp_P \left[B \left(0, \frac{\pi}{\sqrt{R_{\max}/2}} \right) \right] \right) \geq C \min \left\{ \frac{\pi}{pR_{\max}}, \frac{\pi}{qR_{\max}} \right\},$$

where C is a constant.

The proof for Theorem 3.2 can be obtained directly from the same arguments in Corollary 2.2 and Theorem 2.7.

On the universal cover H_p^* of $H_p = M - Q$ at P (let $H_Q = M - P$), we know $B_p^{*r} = \exp_P^*(B(0, r\pi/\sqrt{R_{\max}}))$ is strongly convex if $r < 1/2$ (see Theorem 5.14 in [1]). Let $B_p^r = \Pi^*(B_p^{*r})$, where $\Pi^*: H_p^* \rightarrow H_p$ is the p th covering map.

Lemma 3.3. *Any geodesic η in $N = M - B_p^{1/4}$ which starts and ends at ∂N must have length $L[\eta] \geq 3\pi/(2\sqrt{R_{\max}})$.*

Proof. Let γ be the shortest geodesic which starts and ends at ∂N . Then as defined and discussed in Lemma 2.3, we know $\gamma_\alpha \cup \gamma_\beta \cup \gamma$ is a geodesic 1-gon at the orbifold point P . By Lemma 2.1, we have $L[\gamma_\alpha \cup \gamma_\beta \cup \gamma] \geq 2\pi/\sqrt{R_{\max}}$. Since $\alpha, \beta \in \partial B_p^{1/4}$,

$$(3.9) \quad L[\gamma_\alpha] = L[\gamma_\beta] = \frac{\pi}{4\sqrt{R_{\max}}}.$$

Thus we have $L[\gamma] \geq 3\pi/(2\sqrt{R_{\max}})$.

Lemma 3.4. *Let γ be any geodesic 1-gon in the closure of $B_p^{1/4} - B_p^{1/8}$ with the endpoint on $\partial B_p^{1/4}$. Then we have $L[\gamma] \geq C_1/\sqrt{R_{\max}}$, where*

$$(3.10) \quad C_1 = \min \left\{ 2\pi, \frac{2\sqrt{2}\pi}{p} \sin \left(\frac{\sqrt{2}\pi}{16} \right) \right\}.$$

Proof. Since we have $d(P, Q) \geq \pi/\sqrt{R_{\max}}$ and $i(P, H_p^*) \geq \pi/\sqrt{R_{\max}}$, all the arguments given in Lemma 2.4 can be applied directly.

Lemma 3.5. *If γ is the shortest geodesic 2-gon in M with at least one endpoint in $N_{PQ} = M - B_p^{1/4} - B_Q^{1/4}$, then $L[\gamma] \geq (C_{pq})/\sqrt{R_{\max}}$, where*

$$(3.11) \quad C_{PQ} = \min \left\{ 2\pi, \frac{3\pi}{2}, \pi, \frac{2\sqrt{2}\pi}{p} \sin \left(\frac{\sqrt{2}\pi}{16} \right), \frac{2\sqrt{2}\pi}{q} \sin \left(\frac{\sqrt{2}\pi}{16} \right), \frac{\pi}{4} \right\}.$$

Proof. If γ makes angles at two points, call them A and B . If γ makes an angle at A , choose B to divide γ in half. If γ makes no

angles, choose A and B again to divide γ in half. Let γ be the union of the two geodesic segments γ_1 and γ_2 .

Since $C_{PQ} \leq \pi/4$, we may assume that $L[\gamma] \leq \pi/(4\sqrt{R_{\max}})$.

Case 1. When $A \in \text{int}(N_{PQ})$.

This implies $\angle A = \pi$. Any closed curve which connects $\partial B_P^{1/4}$ and $\partial B_P^{1/8}$ (or $\partial B_Q^{1/4}$ and $\partial B_Q^{1/8}$) has length $L \geq \pi/(4\sqrt{R_{\max}})$. Since $C \leq \pi/4$, we may let $B \in \text{int}(M - B_P^{1/4} - B_Q^{1/4})$. By Lemma 5.6 in [1], we have $\angle B = \pi$. So γ is a closed geodesic. Hence, by Case 1 in Lemma 2.1, we have $L[\gamma] \geq (2\pi)/\sqrt{R_{\max}}$.

Case 2. When $A \in \partial N_{PQ}$ and $B \notin \text{int}(N_{PQ})$.

(I) Suppose $B \notin \partial N_{PQ} = \partial B_P^{1/4} \cup \partial B_Q^{1/4}$, which means $B \in \text{int}(B_P^{1/4} - B_P^{1/8})$ or $\text{int}(B_Q^{1/4} - B_Q^{1/8})$ since $C \leq \pi/4$. So $\angle B = \pi$ and γ is a geodesic 1-gon. The shortest distance between $\partial B_P^{1/4}$ and $\partial B_Q^{1/4}$ is larger than $(\pi/2)\sqrt{R_{\max}}$ and $C \leq \pi/4$. Thus if $A \in \partial B_P^{1/4}$, then $B \in \text{int}(B_P^{1/4} - B_P^{1/8})$, and if $A \in B_Q^{1/4}$, then $B \in \text{int}(B_Q^{1/4} - \partial B_Q^{1/8})$. Hence by Lemmas 3.3 and 3.4, we have $L[\gamma] \geq C_2/\sqrt{R_{\max}}$, where

$$(3.12) \quad C_2 = \min \left\{ 2\pi, \frac{2\sqrt{2}\pi}{p} \sin \left(\frac{\sqrt{2}\pi}{16} \right), \frac{2\sqrt{2}\pi}{q} \sin \left(\frac{\sqrt{2}\pi}{16} \right) \right\}.$$

(II) Suppose $B \in \partial B_P^{1/4} \cup \partial B_Q^{1/4}$. Assume that A and B are both on the same connected component of the boundary, namely $\partial B_P^{1/4}$. Then if γ intersects $N_P = M - B_P^{1/4}$, we will get a geodesic segment η from γ , which starts and ends at ∂N_P . By Lemma 3.3, we have $L[\gamma] \geq L[\eta] \geq 3\pi/(2\sqrt{R_{\max}})$.

If γ does not intersect $N_P = M - B_P^{1/4}$, then γ_1 and γ_2 are both in $B_P^{1/4}$. So we have $\angle A \leq \pi$ and $\angle B \leq \pi$. By Lemma 2.1, we have $L[\gamma] \geq 2\pi/\sqrt{R_{\max}}$.

Assume that A and B are not on the same connected component of the boundary, namely A on $\partial B_P^{1/4}$ and B on $\partial B_Q^{1/4}$. From Theorem 3.1, we have

$$(3.13) \quad d(A, B) + \frac{\pi}{2\sqrt{R_{\max}}} \geq d(P, Q) \geq \frac{\pi}{\sqrt{R_{\max}}},$$

which implies that $L[\gamma] \geq 2 \cdot d(A, B) \geq 2\pi/\sqrt{R_{\max}}$.

Corollary 3.6. For each point $\xi \in$ the closure of N_{PQ} , we have

$$(3.14) \quad i(\xi, M) \geq \frac{C_{pq}}{2\sqrt{R_{\max}}},$$

where

$$(3.15) \quad C_{pq} = \min \left\{ 2\pi, \frac{3\pi}{2}, \frac{2\sqrt{2}\pi}{p} \sin \left(\frac{\sqrt{2}\pi}{16} \right), \frac{2\sqrt{2}\pi}{q} \sin \left(\frac{\sqrt{2}\pi}{16} \right), \frac{\pi}{4} \right\}.$$

Theorem 3.7. *If M is any football with positive curvature, then for each point $\xi \in M$, we have*

$$(3.16) \quad A(\exp_{\xi}(B(0, \pi / \sqrt{R_{\max}/2}))) \geq C/R_{\max},$$

where C is a constant.

Proof. If $\xi \in$ the closure of N_{PQ} , by Corollary 3.6, we then have

$$(3.17) \quad \begin{aligned} A(\exp_{\xi}(B(0, \pi / \sqrt{R_{\max}/2}))) &\geq A(\exp_{\xi}(B(0, C_{PQ} / \sqrt{R_{\max}/2}))) \\ &\geq A(\exp_{\xi}(B(0, \frac{1}{2}C_{PQ} / \sqrt{R_{\max}/2}))) \\ &\geq 2(1 - \cos C_{PQ})[C_{PQ}]^2 \frac{1}{4}\pi/R_{\max}. \end{aligned}$$

If $\xi \in B_P^{1/4}$ (or $\in B_Q^{1/4}$), then

$$(3.18) \quad \exp_P(B(0, \frac{1}{4} / \sqrt{R_{\max}/2})) \subset \exp_{\xi}(B(0, \frac{1}{2} / \sqrt{R_{\max}/2})),$$

$$(3.19) \quad A(\exp_P^*(B(0, \frac{1}{4} / \sqrt{R_{\max}/2}))) \geq \frac{\pi}{8R_{\max}},$$

and

$$(3.20) \quad \begin{aligned} A(\exp_{\xi}(B(0, \pi / \sqrt{R_{\max}/2}))) &\geq A(\exp_{\xi}(B(0, \frac{1}{2} / \sqrt{R_{\max}/2}))) \\ &\geq A(\exp_{\xi}(B(0, \frac{1}{4} / \sqrt{R_{\max}/2}))) \\ &\geq \frac{2\pi(1 - \cos 1/4)}{8pR_{\max}}. \end{aligned}$$

Let

$$(3.21) \quad C = \min \left\{ 2(1 - \cos C_{PQ})[C_{PQ}]^2, \frac{2\pi(1 - \cos 1/4)}{8qR_{\max}}, \frac{2\pi(1 - \cos 1/4)}{8pR_{\max}} \right\}.$$

4. Ricci solitons on bad orbifolds

We know that the Ricci flow is the gradient flow of the relative energy:

$$E(g, h) = \int_M \log(g/h)(R_g \mu_g + R_h \mu_h),$$

$$(4.1) \quad \frac{\partial E}{\partial t} = -2 \int_M (R_q - r)^2 \mu_g.$$

The relative energy E has no absolute minimum on bad orbifolds because there are no metrics of constant curvature on bad orbifolds. (See Figure 8.) There is a 1-parameter family of conformal diffeomorphisms of the orbifold to itself, along which the soliton flows. Translating any metric by one of these diffeomorphisms reduces its energy by a fixed amount. This is related to Futaki's obstructions (see [2], Theorem 8.3.2, p. 125). For completeness, we would like to discuss the existence and uniqueness of the soliton solution.

Theorem 4.1. *On any 2-dimensional orbifold, there exists a unique Ricci soliton.*

Proof. Richard Hamilton [3, Theorem 10.1] proved that, by providing a way of getting soliton solutions on any orbifolds, the only soliton solution on any 2-dimensional manifold is the metric of constant curvature. Good orbifolds are those which can be covered by manifolds, where the group action induced by the covering map is an isometry. Under the Ricci flow any isometry is preserved. So the uniqueness of the Ricci soliton solution on a good orbifold can be obtained directly from the cover manifold. Here we are going to show the existence and uniqueness of the soliton solution on any bad orbifold. We warn the reader that the notation we use here may differ from that in [3]. From [3], we know that the way to get a soliton solution on a (Z_α, Z_β) -football, where $\alpha < \beta$, is to find a constant $k \in (0, 1)$ such that the equation $y = ke^{y-1}$ has two solutions

$$(4.2) \quad \begin{cases} y = 1 - p < 1, \\ y = 1 + q > 1, \end{cases}$$

and

$$(4.3) \quad 0 < \alpha/\beta = p/1 < 1.$$

For any given α and β , if there is a Ricci soliton, then there exists (at least) a constant $b \in (0, \frac{1}{\alpha})$, such that

$$(4.4) \quad \begin{cases} p = b \cdot \alpha, \\ q = b \cdot \beta. \end{cases}$$

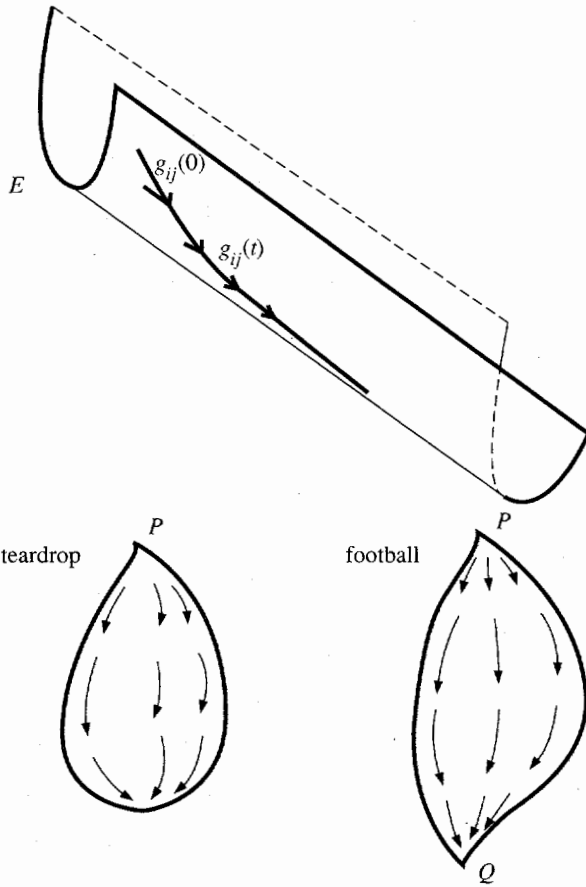


FIGURE 8

So

$$(4.5) \quad \begin{cases} y = 1 - ba < 1, \\ y = 1 + b\beta > 1. \end{cases}$$

Define $f: (0, \frac{1}{\alpha}) \rightarrow R$ by

$$(4.6) \quad f(b) \equiv \frac{1 + b\beta}{1 - b\alpha} - e^{b(\alpha + \beta)}.$$

Note that if $k \in (0, 1)$, then

$$\begin{aligned}
 (4.7) \quad & \Leftrightarrow y = ke^{y-1} \text{ has two solutions } \begin{cases} y = 1 - b\alpha < 1, \\ y = 1 + b\beta > 1. \end{cases} \\
 & \Leftrightarrow 1 - b\alpha = ke^{-b\alpha} \text{ and } 1 + b\beta = ke^{b\beta}, \quad \text{for } b \in (0, \frac{1}{\alpha}); \\
 & \Leftrightarrow \frac{1 - b\alpha}{e^{-b\alpha}} = \frac{1 + b\beta}{e^{b\beta}} = k < 1, \quad \text{for } b \in (0, \frac{1}{\alpha}). \\
 & \Leftrightarrow \frac{1 + b\beta}{1 - b\alpha} = e^{b(\alpha+\beta)}, \quad \text{for } b \in (0, \frac{1}{\alpha}); \\
 (4.8) \quad & \Leftrightarrow f(b) = 0, \quad \text{for } b \in (0, \frac{1}{\alpha}).
 \end{aligned}$$

$$\begin{aligned}
 (4.9) \quad & f^{(1)}(b) = (\alpha + \beta) \left[\frac{1}{(1 - b \cdot \alpha)^2} - e^{b(\alpha+\beta)} \right], \\
 & f^{(2)}(b) = (\alpha + \beta) \left[\frac{2\alpha}{(b - b \cdot \alpha)^3} - (\alpha + \beta)e^{b(\alpha+\beta)} \right].
 \end{aligned}$$

Particularly, we have

$$\begin{aligned}
 (4.10) \quad & f(0) = 0, \\
 & f^{(1)}(0) = 0, \\
 & f^{(2)}(0) = (\alpha + \beta)(\alpha - \beta) < 0,
 \end{aligned}$$

and $\lim_{b \rightarrow 1/\alpha} f(b) = +\infty$. So there exists a smallest nonzero $\bar{b} \in (0, \frac{1}{\alpha})$ such that $f(\bar{b}) < 0$ is a local minimum. On the other hand,

$$\begin{aligned}
 (4.11) \quad & f^{(1)}(\bar{b}) = (\alpha + \beta) \left[\frac{1}{(1 - \bar{b} \cdot \alpha)^2} - e^{\bar{b}(\alpha+\beta)} \right] = 0. \\
 & f(\bar{b}) = \frac{1 + \bar{b} \cdot \beta}{1 - \bar{b} \cdot \alpha} - e^{\bar{b}(\alpha+\beta)} < 0. \\
 & \frac{1 + \bar{b} \cdot \beta}{1 - \bar{b} \cdot \alpha} - \frac{1}{(1 - \bar{b} \cdot \alpha)^2} < 0, \\
 & (1 + \bar{b} \cdot \beta)(1 - \bar{b} \cdot \alpha) - 1 < 0, \\
 & \beta - \alpha - \bar{b} \cdot \alpha\beta < 0.
 \end{aligned}$$

Hence $\forall b > \bar{b}$, $\beta - \alpha - b \cdot \alpha\beta < 0$.

Claim 1. For any solution \tilde{b} of $f(b) = 0$ between 0 and $\frac{1}{\alpha}$,

$$(4.12) \quad f^{(1)}(\tilde{b}) > 0.$$

Suppose $\tilde{b} > 0$ and $f(\tilde{b}) = ((1 + \tilde{b} \cdot \beta)/(1 - \tilde{b} \cdot \alpha)) - e^{\tilde{b}(\alpha+\beta)} = 0$. Since $f(b)$ is nonpositive near $b = 0$ and \bar{b} is the smallest local minimum, we

have $\tilde{b} > \bar{b}$. So

$$(4.13) \quad \beta - \alpha - \tilde{b} \cdot \alpha\beta < 0,$$

and

$$(4.14) \quad \begin{aligned} f^{(1)}(\tilde{b}) &= (\alpha + \beta) \left[\frac{1}{(1 - \tilde{b} \cdot \alpha)^2} - e^{\tilde{b} \cdot (\alpha + \beta)} \right] \\ &= (\alpha + \beta) \left[\frac{1}{(1 - \tilde{b} \cdot \alpha)^2} - \frac{1 + \tilde{b} \cdot \beta}{1 - \tilde{b} \cdot \alpha} \right] \\ &= \tilde{b}(\alpha + \beta) \left[\frac{\tilde{b}\alpha\beta + \alpha - \beta}{(1 - \tilde{b} \cdot \alpha)^2} \right] > 0. \end{aligned}$$

This proves Claim 1. Claim 1 also tells us that we have one and only nonzero solution \tilde{b} for $f(b) = 0$ between 0 and $\frac{1}{\alpha}$. That implies the existence and uniqueness of the soliton solution on a 2-dimensional bad orbifold.

Corollary 4.2. *The Ricci flow gives a canonical metric for any 2-dimensional orbifold.*

On any orbifold with $\chi > 0$, the soliton solution is rotationally symmetric metric of positive curvature. That assures that the soliton solution can be embedded in R^3 . To see the exact shape of the soliton solution of a (Z_α, Z_β) -football in R^3 , one has to explicitly solve the inverse function $y = h^{-1}(u)$ of

$$(4.15) \quad u = \int_{\frac{y_1+y_2}{2}}^y \frac{dy}{y - ke^{y-1}} = h(y), \quad y_1 \leq y \leq y_2,$$

where $k \in (0, 1)$, y_1 and y_2 are the two solutions of $y - ke^{y-1} = 0$, and

$$(4.16) \quad \frac{1 - y_1}{y_2 - 1} = \frac{\alpha}{\beta}.$$

($\forall y_1 < y < y_2, y - ke^{y-1} > 0$, so h^{-1} is well defined.) (See Theorem 10.1 in [3].) The soliton solution on a (Z_α, Z_β) -football induces a metric $ds^2 = g(x)(dx^2 + d\theta^2)$ on the cylinder with coordinates (x, θ) , where

$$(4.17) \quad \begin{aligned} g(x) &= g(x + Ct) \\ &= \frac{r}{C^2} [y - ke^{y-1}], \\ u &= \frac{Cx}{r}, \end{aligned}$$

and C is the velocity with which the soliton is moving by translation in x . Look at the embedding i^* from the cylinder to the space R^3 , using cylindrical coordinates (f, θ, z) :

$$(4.18) \quad i^*: (x, \theta) \rightarrow (f(z), \theta, z(x)).$$

The induced metric on the embedded orbifold is

$$ds^2 = (1 + f_z^2)dz^2 + f^2 d\theta^2,$$

where

$$\left(\frac{dz}{dx}\right)^2 = \frac{4g^2 - g_x^2}{4g} = \frac{1}{\sqrt{\left(\frac{2g}{g_x}\right)^2 - 1}},$$

$$(4.19) \quad \frac{g_x}{g} = \frac{r}{C}(1 - ke^{y-1}).$$

In particular, we have

$$(4.20) \quad \frac{\left|\frac{2g}{g_x}(-\infty)\right|}{\left|\frac{2g}{g_x}(+\infty)\right|} = \frac{\frac{r}{C}(ke^{y_2-1} - 1)(+\infty)}{\frac{r}{C}(1 - ke^{y_1-1})(-\infty)} \\ = \frac{ke^{y_2-1} - 1}{1 - ke^{y_1-1}} = \frac{y_2 - 1}{1 - y_1} = \frac{\beta}{\alpha},$$

which implies that by choosing the constant C suitably, we will have $f_z(z(+\infty)) = 1/\sqrt{\alpha^2 - 1}$ and $f_z(z(-\infty)) = 1/\sqrt{\beta^2 - 1}$. Hence the angles at the two endpoints are $2\pi/\alpha$ and $2\pi/\beta$, and the embedding defined above is a (Z_α, Z_β) -football in R^3 .

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